International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

From Basic Logic to Quantum Logics with Cut-Elimination

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Received July 4, 1997

The results presented in this paper were obtained in the framework of basic logic, a new logic aiming at the unification of several logical systems. The first result is a sequent formulation for orthologic which allows the use of methods of proof theory in quantum logic. Such a formulation admits a very simple procedure of cut-elimination and hence, because of the subformula property, also a method of proof search and an effective decision procedure. By using the framework of basic logic, we also obtain a cut-free formulation for orthologic with implication, for linear orthologic, and, more in generally for a wide range of new quantum-like logics. These logics meet some requirements expressed by physicists and computer scientists. In particular, we propose a good candidate for a linear quantum logic with implication.

1. INTRODUCTION

A sequent calculus for quantum logic was introduced 20 years ago by Dummett (1976) and Nishimura (1980); it was soon after developed by Cutland and Gibbins (1982). Later contributions are by Tamura (1988), Nishimura (1994), and Takano (1995).

Our results were obtained in a different framework, namely that of basic logic. The first version of basic logic was introduced by Battilotti and Sambin (1996) as a common denominator of classical, intuitionistic, and linear logic, and of orthologic. The present formulation of basic logic, as developed in Battilotti, *et al.* (1996), enjoys a few quite strongly desirable properties: symmetry of the calculus, cut elimination, and hence in general a good proof theory. The price, however, is that orthologic was no longer among its extensions. To recapture orthologic, it was necessary to add negation. As has

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0020-7748/98/0100-003 1$15.00/0

been shown by the first author, if negation in Girard's style is added to basic logic, a new approach is possible, leading in particular to a new sequent formulation for orthologic which admits cut-elimination (Faggian, 1996).

2. A CUT-FREE CALCULUS FOR ORTHOLOGIC . . .

The starting point is the sequent calculus **BS** for basic logic supplemented with structural rules; its fragment on the language with connectives $\wedge$ and $\vee$ is the following (where we assume Γ, Δ, Σ, Λ to be finite sets of formulas):

BS^{-}

Axioms

$$A \vdash A$$

Rules on Connectives

$$\frac{A \vdash \Delta}{A \wedge B \vdash \Delta} \quad \frac{B \vdash \Delta}{A \wedge B \vdash \Delta} \wedge L \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R$$

$$\frac{A \vdash \Delta \quad B \vdash \Delta}{A \vee B + \Delta} \vee L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee R$$

Structural Rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma \vdash \Lambda, \Delta} \quad \textit{weakening}$$

In order to have a quantum logic, two fundamental properties are required: nondistributivity and an involutive negation for which De Morgan's rules hold. The above logic is nondistributive, but the involution is still missing.

The natural solution is to extend the language and to adopt Girard's negation; in this way, negation (which here we call orthogonal) is not a connective, but is defined. The key point is to assume as primitives of the language not only propositional variables, but also their duals. Indeed, the propositional literals are assumed to be given in pairs, one positive (written p) and one negative (written $p^{\perp}$). So:

(i) Atomic formulas are propositional letters $p, q, r, \ldots$ and their duals $p^{\perp}, q^{\perp}, r^{\perp}, \ldots$.

(ii) Formulas are constructed from atomic ones by closing the application of the binary connectives $\wedge$, $\vee$.

Negation of a formula is defined as follows:

$$p^{\perp\perp} \equiv p, \qquad (A \wedge B)^{\perp} \equiv A^{\perp} \vee B^{\perp}, \qquad (A \vee B)^{\perp} \equiv A^{\perp} \wedge B^{\perp}$$

The calculus obtained by adding such orthogonal to $\mathbf{BS}^{-}$ is denoted by $^{\perp}\mathbf{BS}$. It produces a logic which is equivalent to paraconsistent quantum logic introduced by Dalla Chiara and Giuntini (1989) we prefer to call it basic orthologic. It is orthologic without the two laws of *noncontradiction* and *excluded middle.*

Orthologic is obtained by adding such laws expressed through two new structural rules, named *transfer* (1 and 2):

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Delta^{\perp} \vdash}\, tr1 \qquad \frac{\Gamma \vdash \Delta}{\vdash \Gamma^{\perp}, \Delta}\, tr2$$

The resulting calculus, called $^{\perp}\mathbf{O}$, is easily seen to be equivalent to that given by Cutland and Gibbins (1982) if negation $\neg A$ is interpreted into $A^{\perp}$; both calculi are repeated in the Appendix. Note that the two rules of cut are not given in the table of rules for $^{\perp}\mathbf{O}$, since the calculus admits their elimination.

As in Gentzen, the procedure of cut-elimination is obtained by an induction on two parameters: degree and rank of the cut formula. The calculus $^{\perp}\mathbf{O}$ allows us to overcome in a simple way the two problems which make cut-elimination for orthologic difficult: (i) constraints on contexts and (ii) negation.

(i) Recall that nondistributivity of quantum logic is obtained by imposing restrictions on the context of those rules which are needed to prove distributivity. In particular, the rule which introduces $\vee$ on the left (here indicated with $\vee L$) must have empty context on the left. Now consider the derivation

$$\frac{\dfrac{A \vdash C \wedge D \quad B \vdash C \wedge D}{A \vee B \vdash C \wedge D}\,\vee L \qquad \dfrac{\Gamma, C \vdash \Delta}{\Gamma, C \wedge D \vdash \Delta}}{\Gamma, A \vee B \vdash \Delta}\, cut1$$

In this derivation, the cut-formula is principal on the right premiss and hence the right rank is 1. So Gentzen's procedure to lower the rank must operate at the left and would necessarily produce the two derivations

$$\frac{A \vdash C \wedge D \quad \Gamma, C \wedge D \vdash \Delta}{\Gamma, A \vdash \Delta}\, cut1 \qquad \frac{B \vdash C \wedge D \quad \Gamma, C \wedge D \vdash \Delta}{\Gamma, B \vdash \Delta}\, cut1$$

At this point one would like to conclude by applying $\vee L$ and obtain $\Gamma, A \vee B \vdash \Delta$, but this is not allowed unless Γ is empty.

In the present formulation this problem does not arise, because every principal formula has empty context. So the reduction can be applied.

(ii) It is important to recall that the orthogonal is not a connective, but it is defined. So the only rules related to negation are the structural rules of transfer. To reduce the rank in this case, the way out is to exploit symmetry as fully as possible.

Girard's negation has the nice property that every formula A and its dual $A^{\perp}$ have exactly the same degree. The same idea can be extended to derivations, and hence to the rank of a cut in the following way. By the symmetry of the calculus, the rule

$$\frac{\Gamma \vdash \Delta}{\Delta^{\perp} \vdash \Gamma^{\perp}}$$

is derivable together with its inverse. This means that if one has a derivation

$$\begin{array}{c} \vdots\,\Pi \\ \Gamma \vdash M, \Delta \end{array}$$

then one also has (in an immediate and effective way) also the dual derivation

$$\begin{array}{c} \vdots\,\Pi^{\perp} \\ \Delta^{\perp}, \quad M^{\perp} \vdash \Gamma^{\perp} \end{array}$$

The two derivations Π and $\Pi^{\perp}$ have exactly the same height, or better, they have the same (symmetrical) structure. Thus, in particular, if M is principal, $M^{\perp}$ is principal. If M has rank r, then $M^{\perp}$ has the same rank r.

Consider now the reduction for transfer. If the given derivation is of the form

$$\dfrac{\Sigma \vdash M \qquad \dfrac{\begin{array}{c}\vdots\,\Pi \\ \Gamma \vdash M^{\perp}, \Delta\end{array}}{\Gamma, M, \Delta^{\perp} \vdash}\, tr1}{\Gamma, \Sigma, \Delta^{\perp} \vdash}\, cut1$$

then the new trick, called *flipping*, is to consider the dual derivation and thus reduce it to

$$\dfrac{\dfrac{\Sigma \vdash M \qquad \begin{array}{c}\vdots\,\Pi^{\perp} \\ \Delta^{\perp}, M \vdash \Gamma^{\perp}\end{array}}{\Sigma, \Delta^{\perp} \vdash \Gamma^{\perp}}\, cut1}{\Gamma, \Sigma, \Delta^{\perp} \vdash}\, tr1$$

As in Gentzen, the cut-elimination procedure is fully effective. For more details see Faggian (1996).

3. . . . AND A GAMMA OF NEW QUANTUM-LIKE LOGICS

Now we briefly illustrate some relevant consequences of our approach:

a. *Quantum logic with implication.* The starting point to obtain orthologic was a fragment of structural basic logic **BS.** If we add Girard's negation and the transfer rules to the full calculus of **BS**, we obtain a cut-free calculus for orthologic with implication (and anti-implication).

b. *A wide range of quantum-like logics.* We can recover with a new characterization the cube of logics by Battilotti and Sambin and the ideas which inspired it. Thus, as a common denominator of linear logic and (basic) orthologic (with or without implication), we obtain a whole range of new quantum-like logics with a good proof-theoretic formulation, as the following table of logics shows. For all of them we have a sequent calculus and a proof of cut elimination:

Quantum-like, without $\rightarrow$	Quantum-like, with $\rightarrow$
basic orthologic $\mathbf{BS}^-$	**BS**
linear basic orthologic $\mathbf{B}^-$	**B**
orthologic $\mathbf{BS}^- + tr$	$\mathbf{BS} + tr$
linear orthologic $\mathbf{B}^- + tr$	$\mathbf{B} + tr$

Such new logics meet some of the requirements of physics and computer science expressed, for instance, by Dalla Chiara, Giuntini, and Pratt (Pratt, 1993). In particular, the logical system $\mathbf{B} + tr$ is a good candidate to satisfy the requirements for a linear quantum logic with implication.

c. *Proof search and effective decision procedure.* By Gentzen's method, we have a method of proof search and hence an effective decision procedure for provability in orthologic and in all the quantum-like logics here considered.

APPENDIX

The Calculus $^{\perp}$O

Axioms

$$A \vdash A$$

Rules on Connectives

$$\frac{A \vdash \Delta}{A \wedge B \vdash \Delta} \quad \frac{B \vdash \Delta}{A \wedge B \vdash \Delta}\wedge L \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge R$$

$$\frac{A \vdash \Delta \quad B \vdash \Delta}{A \vee B \vdash \Delta}\vee L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vee R$$

Structural Rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma \vdash \Lambda, \Delta}\ \textit{weakening}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Delta^{\perp} \vdash}\ tr1 \qquad \frac{\Gamma \vdash \Delta}{\vdash \Gamma^{\perp}, \Delta}\ tr2$$

Cutland and Gibbins (1982)

Axioms

$$A \vdash A$$

Rules on Connectives

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge \vdash) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\vdash \wedge)^{\dagger}$$

$$\frac{A \vdash \Delta \; B \vdash \Delta}{A \vee B \vdash \Delta}\ (\vee \vdash)^{\dagger} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vdash \vee)^{\dagger}$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash}\ (\neg \vdash)^{\dagger}$$

$$\frac{\Gamma + \Delta}{\neg\Delta \vdash \neg\Gamma}\ (\vdash \neg)^{\dagger}$$

$$\frac{A, \Gamma + \Delta}{\neg\neg A, \Gamma \vdash \Delta}\ (\neg\neg \vdash) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, \neg\neg A}\ (\vdash \neg\neg)$$

Structural Rules

$$\frac{\Gamma \vdash \Delta}{\Theta, \Gamma \vdash \Delta, \Sigma}\ (ext)$$

Cuts

$$\frac{\Gamma \vdash M \quad \overline{\Gamma}, M \vdash \overline{\Delta}}{\Gamma, \overline{\Gamma} \vdash \overline{\Delta}}\ cut1 \qquad \frac{\overline{\Gamma} \vdash M, \overline{\Delta} \quad M \vdash \Delta}{\overline{\Gamma} \vdash \Delta, \overline{\Delta}}\ cut2$$

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